Lecture 12

For the theory of integration (of measurable functions) we shall need the concept of simple functions to approximate general measurable functions.

Special constructions

① If $f: X \to \overline{\mathbb{R}}$, then

$$f^+ = \max(f, 0), \quad f^- = \max(-f, 0),$$

the positive and negative parts of f.
Note $f^+, f^- \geq 0$ and $f = f^+ - f^-$.

② If $E \subseteq X$, the indicator fcn $f: X \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 0, & x \notin E \\ 1, & x \in E. \end{cases}$$

Clearly, for any $a \in \mathbb{R}$, $f^{-1}((a,\infty))$ is either X $(a<0)$, E $(0 \le a < 1)$, $\emptyset$ $(a \ge 1)$ $\Rightarrow$ f $\mathcal{M}$-meas. $\Leftrightarrow$ $E \in \mathcal{M}$.

③ A simple function f is a linear combination of indicator fcns:

$$f = \sum_{k=1}^{n} c_k \chi_{E_k}, \quad c_k \in \mathbb{C} \text{ (or } \mathbb{R})$$

Its standard representation is given as follows, let $G = f(X) \subseteq \mathbb{C}$. f is simple $\Leftrightarrow$ G is finite $\{z_1, \dots, z_n\}$. The SR is then

$$f = \sum_{k=1}^{n} z_k \chi_{f^{-1}(z_k)}.$$

Ex. If $E \subseteq X$, then the SR of $f = \chi_E$ is

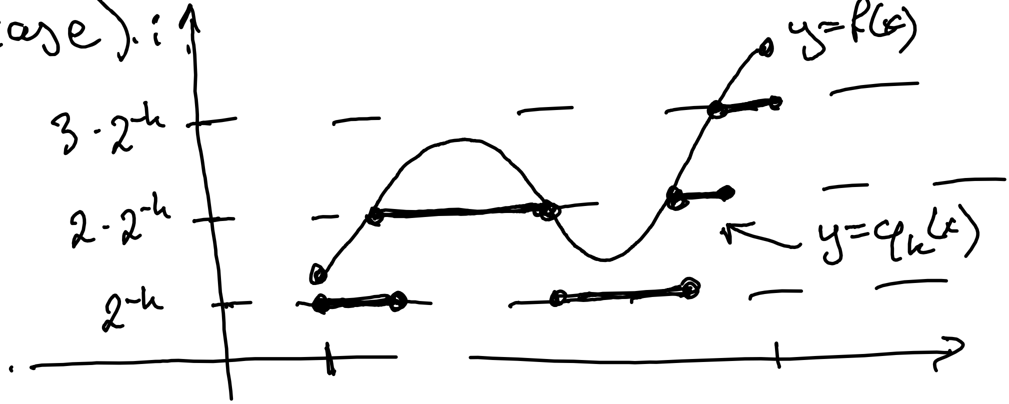
$$f = 0 \cdot \chi_{E^c} + 1 \cdot \chi_E.$$

Thm 1 Let $(X, \mathcal{M})$ be meas. space. If $f: X \to \mathbb{C}$ is measurable, then $\exists$ seq. of simple functions $\{\varphi_n\}_{n=1}^{\infty}$ s.t.

$$0 \le |\varphi_1| \le |\varphi_2| \le \dots, \quad \varphi_n(x) \to f(x), \ \forall x \in X,$$

and $\varphi_n \to f$ uniformly any set where f is bounded.

Pf. It suffices to prove for $f: X \to \overline{\mathbb{R}}$ with $f \ge 0$, by decomposing $f = h + ig$ and $h = h^+ - h^-$, $g = g^+ - g^-$. Thus, WLOG assume $f \ge 0$. Idea is simple: decompose the range $[0, \infty]$ into dyadic intervals (as opposed to the domain in the Riemann integral case).:



Thus, for $n \geq 1$, consider the dyadic decomposition

$$[0, 2^n) = \bigcup_{k=1}^{2^{2n}} \underbrace{[(k-1)2^{-n}, k2^{-n})}_{I_{k,n}}$$

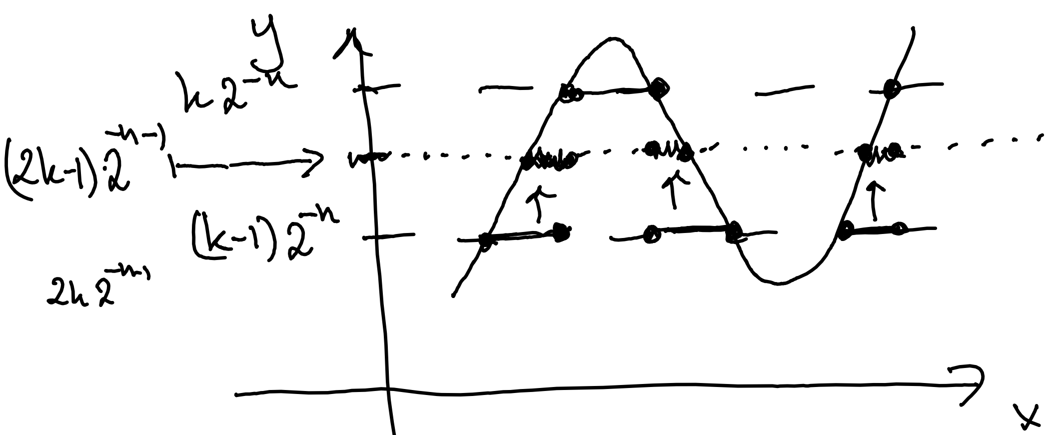
and let

$$\varphi_n(x) = \sum_{k=1}^{2^{2n}} (k-1)2^{-n} \chi_{f^{-1}(I_{k,n})} + 2^n \chi_{f^{-1}([2^n, \infty])}$$

- If $E = \{f(x) \leq M\}$ and $M < 2^N$, then $E = \bigcup_{k=1}^{2^{2n}} f^{-1}(I_{k,n})$ and $\forall x \in E$

$$|f(x) - \varphi_n(x)| < 2^{-n}, \quad n \geq N.$$

- $0 \leq \varphi_1 \leq \varphi_2 \leq \ldots$. "Proof by Pic":



- Clearly, $\varphi_n(x) \nearrow f(x)$, $\forall x$. $\boxtimes$